Exercise Sheet 12

Exercise 1. Let M be a smooth manifold of real dimension m, and let $(x^1, \dots, x^m)$ be local coordinates on an open set $U \subset M$. Any k-form on U can be written as

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le m} \omega_{i_1 \cdots i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \qquad \omega_{i_1 \cdots i_k} \in C^\infty(U).$$

In these coordinates, the exterior derivative is given by

$$d\omega = \sum_{1 \le i_1 < \cdots < i_k \le m} \sum_{j=1}^{m} \frac{\partial \omega_{i_1 \cdots i_k}}{\partial x^j} \, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Use this formula to verify the defining properties of the exterior derivative $d : \Omega^k(U) \to \Omega^{k+1}(U)$:

(1) For $f \in \Omega^0(U) = C^\infty(U)$, the form df is the differential of f.
(2) For all $f \in \Omega^0(U)$ we have $d(df) = 0$.
(3) (Leibniz rule) For $\alpha \in \Omega^p(U)$ and $\beta \in \Omega^q(U)$ we have

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \, \alpha \wedge d\beta.$$

(4) For every differential form $\omega \in \Omega^k(U)$ we have $d(d\omega) = 0$.

Exercise 2. Let $U \subset \mathbb{R}^2$ be a contractible open set. Show that every closed differential 1-form on U is exact.

Exercise 3. (for credit, due on 21 December)
Let X be a Riemann surface and (U, z) a chart of X. Recall that the $*$ operator on 1-forms and the Laplace operator Δ taking 0-forms to 2-forms are defined by

$$*\omega = -i\omega^{(1,0)} + i\omega^{(0,1)},$$
$$\Delta f = -d * df.$$

We say that $\eta \in \Omega^1(U)$ is harmonic if it is locally of the form $\eta = df$ for $f \in \Omega^0(U)$ with $\Delta f = 0$.

(1) *(1 point)* Show that a 1-form $\eta \in \Omega^1(U)$ is harmonic if and only if $d\eta = 0$ and $d(*\eta) = 0$. **Hint:** Use the Poincaré lemma (Exercise 2).
(2) *(2 points)* Show that a 1-form $\eta \in \Omega^1(U)$ is harmonic if and only if it is of the form $\eta = \omega_1 + \overline{\omega_2}$ for some holomorphic 1-forms ω_1 and ω_2 on U.
(3) *(1 point)* Show that a 1-form $\omega \in \Omega^1(U)$ is holomorphic if and only if it is of the form $\omega = \eta + i * \eta$ for some harmonic 1-form η on U.
(4) *(1 point)* Show that every holomorphic 1-form on U is closed and harmonic.

Exercise 4. Let X be a compact, connected Riemann surface of genus $g \ge 1$. Let P_g be its planar diagram (without glueing along edges). We denote by P_g° and ∂P_g the interior and boundary of P_g. Let $\{a_i, b_i, \ 1 \le i \le g\}$ be a canonical basis of cycles of $H_1(X, \mathbb{Z})$. For any closed 1-form α, we define its periods

$$A_i(\alpha) = \int_{a_i} \alpha, \quad B_i(\alpha) = \int_{b_i} \alpha$$

with respect to these cycles. The goal of the exercise is to prove Riemann's bilinear identity: If α, β are closed 1-forms, then

$$\int_X \alpha \wedge \beta = \sum_{i=1}^{g} \Big(A_i(\alpha) B_i(\beta) - B_i(\alpha) A_i(\beta) \Big).$$

(1) Explain why $\int_X \alpha \wedge \beta = \int_{P_g^\circ} \alpha \wedge \beta$.

(2) Show that there is a function $f \in C^\infty(P_g^\circ, \mathbb{C})$ such that $\int_X \alpha \wedge \beta = \int_{\partial P_g} f\beta$.
(3) Deduce Riemann's bilinear identity from the previous formula.